Note: From Dirac Equation to the Quest for Majorana Fermions

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The Dirac equation, born out of the mind of Paul Dirac as he was staring at burning flame [1], is one of the greatest cornerstones in theoretical physics, as it predicted the existence of antimatter. Hidden in the equation, which was purportedly remarked by Dirac to be more intelligent than himself, is the possibility that there could exist in Nature, as yet undiscovered fermions, now called *Majorana fermions*, which are antiparticles of themselves. A young promising Italian physicist Ettore Majorana, having peered into the dark secret of Nature, vanished without a trace in 1938 during a trip from Palermo to Naples onboard a ship... This short note explains the beautiful idea behind Majorana's discovery and the current ongoing quest to find Majorana fermions. We touch upon the possible applications of Majorana fermions to quantum computing.

I. THE DIRAC EQUATION REVISITED

We recall that the idea behind Dirac equation is fairly simple. Starting with the relativistic energy-momentum relation $E^2 - \vec{p}^2c^2 = m^2c^4$, or equivalently in the Einstein summation convention of 4-vector formulation $p^\mu p_\mu - m^2c^2 = 0$, Dirac considered factoring this equation into two pieces:

$$p^\mu p_\mu - m^2c^2 = (\beta^\kappa p_\kappa + mc)(\gamma^\kappa p_\kappa - mc), \tag{1}$$

where β^κ and γ^κ are eight coefficients to be determined. Multiplying out the RHS and comparing with LHS, we must have

$$\beta^\kappa = \gamma^\kappa, \quad p^\mu p_\mu = \gamma^\kappa\gamma^\lambda p_\kappa p_\lambda. \tag{2}$$

The second equation can be written out in full and compares with the LHS $(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$, and one finds that to get rid of the cross terms, we need to impose the following conditions:

$$\begin{cases} (\gamma^0)^2 = 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \\ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \;\text{ for } \mu \neq \nu. \end{cases} \tag{3}$$

Equivalently, the requirement is that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{4}$$

where $\{A, B\} = AB + BA$ is the anti-commutator and $\eta_{\mu\nu}$ is the usual Minkowski metric diag$(1, -1, -1, -1)$ on $\mathbb{R}^{3,1}$. This requirement cannot be satisfied by numbers, but they *can* be satisfied by matrices, now called the *gamma matrices*. Dirac found the corresponding matrices in 1928 [2]. They are

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

and

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{6}$$

Conventionally, we choose to describe the physics using the second factor of Eq.(1): $\gamma^\lambda p_\lambda - mc = 0$. Upon the canonical substitution in quantum mechanics $p_\mu \to i\hbar\partial_\mu$, we obtain the celebrated Dirac equation:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\,\psi = 0.$$

It turns out that the defining relation of gamma matrices Eq.(4), actually defines an algebraic structure that was already discovered in 1878 by the mathematician William Kingdon Clifford, who was also the first to suggest that gravitation might be a manifestation of an underlying geometry.

Definition: The *Clifford Algebra* C_k is an associative algebra over $\mathbb{R}$ with multiplicative identity with k generators $e_1, \cdots, e_k$ satisfying $e_i^2 = -1$ for all $i = 1, \cdots, k$ and $e_ie_j = -e_je_i$ for all $i \neq j$.

That is to say, we start with a multiplicative identity, and introduce a bunch of square roots of -1.

We can generalize C_k to $C_{p,q}$, the algebra generated by the e_i's whose p of them are square roots of -1 and q of them are square roots of 1. The gamma matrices are then nothing but the generators of $C_{3,1}$.

The Dirac equation gives a good description to relativistic spin-1/2 particles like the electron and its antiparticle, the positron. The reason why the Dirac's choice of gamma matrices are very useful for this purpose is that the entries of these matrices consist of both real numbers ± 1 and imaginary numbers $\pm i$, i.e., the corresponding quantum field ψ [the *Dirac spinor*] is complex, and thus is natural to describe charged particles. This is because in quantum field theory, if a given field ψ creates a particle A [and annihilates $\bar{A}$, the antiparticle], then its complex conjugate field ψ^* creates the antiparticle $\bar{A}$ [and annihilates A].

Of course, it is well known that photon [spin-1] is its own antiparticle, and so are graviton [spin-2] and neutral pions [spin-0]. These are bosons. The question is whether there exists any fermion that is its own antiparticle.

II. MAJORANA FERMIONS

It turns out that Dirac's choice of gamma matrices are *not* the only set that satisfies the Dirac equation. Recall that:

Definition: A *representation* of an algebra is an algebra homomorphism

$$\rho : A \to \mathrm{End}(V), \tag{7}$$

where V is a vector space and $\mathrm{End}(V)$ is the algebra of all linear maps $T : V \to V$.

Dirac's choice of gamma matrices is called the *Dirac representation* of the Clifford algebra $C_{3,1}$. Indeed, we have the following important result:

Theorem: If γ^μ and $\gamma^{\mu'}$ are two irreducible representations [i.e. there is no nontrivial subrepresentations] of the same Clifford algebra, then there always exists a nonsingular matrix S such that $\gamma^{\mu'} = S\gamma^\mu S^{-1}$. Furthermore, the matrix S is unique up to scalar multiplication.

Central to the proof of this theorem [of which we shall omit], is the well known result from Linear Algebra:

Theorem [Schur's Lemma]: Suppose γ^μ and $\gamma^{\mu'}$ are two irreducible representations of degree n and n' respectively where $n \leqslant n'$, and let S be a matrix with n' rows and n columns such that $\gamma^{\mu'} S = S\gamma^\mu$, then S is either the null matrix or it is nonsingular with $n = n'$.

The choice of representation found by Majorana has the special property that all of the gamma matrices have imaginary entries, this leads to Dirac equation that governs *real* fields, i.e. $\psi = \psi^*$, so that the Majorana fermions are their own antiparticles. The Majorana representation is, explicitly,

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \tag{8}$$

and

$$\gamma^2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

III. PROPERTIES OF DIRAC FERMIONS

We will now look into the details. Much of this section is based on [3]. We first recall that in a quantum theory, the charge conjugation operation $\mathcal{C}$ acts on the gauge field $A_\mu(x)$ and its associated current $J_\mu(x)$ by

$$\begin{cases} A_\mu(x) \xrightarrow{\mathcal{C}} A^{\mathcal{C}}_\mu(x) = \eta_A A_\mu(x) = -A_\mu(x), \\ J_\mu(x) \xrightarrow{\mathcal{C}} J^{\mathcal{C}}_\mu(x) = \eta_J J_\mu(x) = -J_\mu(x). \end{cases} \tag{10}$$

Note that since opposite charges are assigned to particles and antiparticles, charge conjugation really interchanges particles and antiparticles! We now consider a system of Dirac fermions interacting with an electromagnetic field. For said fermion minimally coupled to the photons, we have

$$\left[i\gamma^\mu \left(\partial_\mu + ieA_\mu\right) - m\right]\psi(x) = 0. \tag{11}$$

The adjoint equation is

$$\bar\psi(x)\left[i\gamma^\mu\left(\overleftarrow{\partial}_\mu - ieA_\mu\right) + m\right] = 0 \tag{12}$$

That is, with T denoting transpose operation, we have

$$\left[i(\gamma^\mu)^T\left(\partial_\mu - ieA_\mu\right) + m\right]\bar\psi(x)^T = 0. \tag{13}$$

The positron, identified with the charge conjugated field $\psi^{\mathcal{C}}$, satisfies

$$\left[i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right]\psi^{\mathcal{C}}(x) = 0. \tag{14}$$

Since $-(\gamma^\mu)^T$ satisfies the Clifford algebra, again by the theorem in the preceding section, there exists similarity transformation that relates $-(\gamma^\mu)^T$ with γ^μ. Therefore, there exists a nonsingular matrix C such that

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \Longleftrightarrow C^{-1}\gamma^\mu C = -(\gamma^\mu)^T. \tag{15}$$

Then, from Eq.(13), we would have

$$C\left[i(\gamma^\mu)^T(\partial_\mu - ieA_\mu) + m\right]\bar\psi^T = 0. \tag{16}$$

That is,

$$\left[iC(\gamma^\mu)^T C^{-1}(\partial_\mu - ieA_\mu) + m\right]C\bar\psi^T = 0, \tag{17}$$

or

$$\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]C\bar\psi^T = 0. \tag{18}$$

Upon comparing with Eq.(14), we get

$$\psi^{\mathcal{C}}(x) = \eta_\psi C\bar\psi^T, \quad |\eta_\psi|^2 = 1. \tag{19}$$

A free Dirac particle satisfies the Dirac equation and thus from Eq.(14), so must its antiparticle $\psi^{\mathcal{C}} = \eta_\psi \mathcal{C}\bar\psi^T$. Majorana fermion, being a Dirac particle which is its own antiparticle must then satisfy

$$\psi = \psi^{\mathcal{C}} = \eta_\psi \mathcal{C}\bar\psi^T. \tag{20}$$

Due to this constraint, it can be shown that there is only two independent degrees of freedom. Note that the Majorana fermions can still be massive.

A. Charge Neutrality

From the definition of the antisymmetrized current

$$J^\mu_{\text{anti}}(x) = \frac{1}{2}\left[\bar\psi(x)\gamma^\mu\psi(x) - \psi^T(x)\gamma^{\mu T}\bar\psi^T(x)\right] \tag{21}$$

$$= \frac{1}{2}\left[\bar\psi(x)\gamma^\mu\psi(x) - \bar\psi^{\mathcal{C}}(x)\gamma^\mu\psi^{\mathcal{C}}(x)\right], \tag{22}$$

we have for the Majorana fermions [which satisfies $\psi = \psi^{\mathcal{C}}$], the vanishing of current $J^\mu_{\text{anti}} = 0$. Therefore it follows that the Majorana fermions are charged neutral and cannot have any electromagnetic interaction.

B. Handedness

Recall from elementary particle physics that a particle with definite handedness will lead to an antiparticle with the opposite handedness. Therefore, Majorana fermion, being its own antiparticle, *cannot* have a well defined handedness. That is, we cannot talk of a Majorana particle with a given helicity [This is not true in general spacetime dimension. For example, in $d \equiv 2 \pmod 8$ dimension, we *can* talk about Majorana-Weyl spinors, but we shall not go into the details].

At this point we should discuss whether neutrino can be a Majorana fermion. Neutrino is certainly charged neutral, and in the standard model of particle physics, it has definite handedness due to it being massless. In fact, Majorana himself conjectured the possibility that neutrino could be an example of Majorana fermion. Nevertheless, we now know that neutrinos have very small mass (it is still possible that one of the three neutrinos is massless), and furthermore they can oscillate between various flavors. For neutrinos produced in the the pion decay $\pi^+ \to \mu^+ + \nu$ and the antineutrino produced in similar manner $\pi^- \to \mu^- + \bar{\nu}$, we know that the neutrinos are always left-handed while antineutrinos are always right-handed in the Standard Model. In principle, it is possible that they are still in fact one and the same, only exhibiting different behaviors when in different states of motion, as, due to it having mass, it is always possible to change your frame of reference so that your direction of motion is reversed and thus the handedness is reversed[1]. In any case, if neutrinos are Majorana fermions, lepton number will not be conserved, an effect which has yet to be found in Nature. For the same token, if neutron is a Majorana fermion, baryon number would not conserve. In brief, *the existence of Majorana fermion necessarily violate some kind of fermion number.* We will again come back to this remark when we discuss Majorana mass.

C. Parity

We recall that if the intrinsic parity of a Dirac particle is $\eta_\psi^{\mathcal{P}}$, then the antiparticle will have intrinsic parity $-(\eta_\phi^{\mathcal{P}})^*$. For Majorana fermion,

$$\eta_\psi^{\mathcal{P}} = -(\eta_\phi^{\mathcal{P}})^*. \tag{23}$$

This means that the intrinsic parity for Majorana fermion is *imaginary*!

[1] Some people explain hand-wavingly that neutrino oscillation implies massive neutrino as being a consequence of special relativity – massless particles don't experience time and therefore cannot oscillate. This is more misleading than helpful – neutrinos are produced in flavor eigenstates, not the mass eigenstates.

IV. MAJORANA MASS TERM

There exists another representation of the Clifford algebra $C_{3,1}$ called the *Weyl representation* or *chiral representation.* The Weyl representation is related to the Dirac representation by

$$\gamma^\mu_{\text{Weyl}} = U\gamma^\mu_{\text{Dirac}}U^{-1}, \quad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{24}$$

We remind the readers that here 1 in the matrix is actually a 2×2 identity matrix.

It turns out that the Weyl fermions are irreducible representations of the proper orthochronous Lorentz group, and hence can be used to construct any kind of fermion field [4]. In particular, we can construct Majorana fermion field out of Weyl fields. Let ψ_L be a left-Chiral Weyl field, and thus $(\psi_L)^{\mathcal{C}}$ right-chiral Weyl field. Define the sum of fields $\chi = \psi_L + (\psi_L)^{\mathcal{C}}$. Then $\chi^{\mathcal{C}} = (\psi_L)^{\mathcal{C}} + (\psi_L) = \chi$ and so χ is a Majorana field. One would then wonder, since the two-component Majorana fermion can be massive, can we not also have a mass for the two-component Weyl fermion?

Let us try to write down naively the Lagrangian density for a hypothetical massive left-handed Weyl fermion field:

$$\mathcal{L} = i\bar{\psi_L}\partial\!\!\!/\psi_L - m\bar{\psi_L}\psi_L, \tag{25}$$

where we have employed Feynman's slash notation $A\!\!\!/ := \gamma^\mu A_\mu$ for any 1-form A.

Now, with $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$, we have

$$\bar{\psi_L}\psi_L = \bar{\psi}\frac{1}{2}(1+\gamma^5)\cdot\frac{1}{2}(1-\gamma^5)\psi = 0, \tag{26}$$

so that the mass term trivially vanishes. The Weyl fermion is massless.

We could however consider unconventional Dirac mass term and write down Lagrangian density of the form

$$\mathcal{L} = i\bar{\psi_L}\partial\!\!\!/\psi_L + m\bar{\psi}^{\mathcal{C}}_L\psi_L = i\bar{\psi_L}\partial\!\!\!/\psi_L - m\psi_L^T C^{-1}\psi_L. \tag{27}$$

We have previously remarked that Majorana fermion breaks some kind of fermion number. This will be the case for Majorana mass term. Regardless, Majorana mass term might be useful in providing possible explanation to small masses of neutrinos, which could have arise naturally in grand unified theories in which both baryon and lepton numbers can be violated in small amounts.

V. WHERE ARE THE MAJORANA FERMIONS IN NATURE?

A. Dark Matter and Supersymmerty

In addition to the possibility that neutrino might be Majorana fermions, quests for this realization of the representation of abstract Clifford algebra continues in other arenas of physics. One such possibility is that dark matter, which makes up a quarter or so of the total mass-energy content of the ΛCDM universe, might in fact be Majorana fermions. One popular hypothesis for dark matter is the so-called *weakly interacting massive particle*, or WIMP, of which a possible candidate is one the supersymmetric particles. Supersymmetry, as we recall, is a symmetry that relates bosons to fermions and vice versa. More precisely, a supersymmetric transform turns a bosonic state into a fermionic state and vice versa. This is achieved by extending the Poincaré group to the *super-Poincaré* group by adding two anti-commuting generators Q and $Q^\dagger$ such that

$$\left\{Q, Q^\dagger\right\} = P^\mu, \tag{28}$$

and

$$\{Q, Q\} = \left\{Q^\dagger, Q^\dagger\right\} = 0, \{P^\mu, Q\} = \left\{P^\mu, Q^\dagger\right\} = 0, \tag{29}$$

where P^μ is the four-momentum generator of spacetime translations. Q and $Q^\dagger$ are actually *spinorial* objects (we have suppressed all the spinor indices), this is the reason that the superpartners differ from their SM counterparts in spin.

The SUSY partner of the photon is the *photino*, a spin-1/2 particle. Since the photino mirrors the properties of its partner the photon, it must in particular, be its own antiparticle, and thus a Majorana fermion. Similarly, various other superpartners such as the neutral gauginos and the Higgsinos, are also Majorana fermions; the supersymmetric world is teeming with Majorana fermions! To be a viable candidate for WIMP, we look for the lightest supersymmetric particles, since they are stable and long-lived to survive until the present era. The lightest of all the supersymmetric particle is the *neutralino*, a particle which is not itself an exact partner of any particle in the Standard Model, but a composite of the photinos, Higgsinos and Z-inos. It is also a Majorana fermion. Thus perhaps Majorana fermion is indeed a dark secret of the universe glimpsed by Ettore Majorana... Of course dark matter might be something entirely different; yet SUSY particles [and hence Majorana fermions] could still be found by the LHC [although the parameter space has now been tightly squeezed and yet there is no sight of the SUSY particles...].

B. Solid State Physics and Quantum Computing

Though not real particles, solid state physics provides an exciting arena in looking for the next-best-thing: *quasiparticles* formed from the collective movement of electrons in various materials, such as superconductor. A group of physicists from Delft University of Technology tested a proposal that a pair of Majorana fermions could form at the interface between a superconductor and a semiconducting nanowire in a magnetic field [5]. The idea is that electrical charge in superconductor allows electrons and absences of electrons ["*holes*", which act somewhat like antielectrons] to form neutral entities at the interface with the nano-wire, and hence effectively a Majorana fermion [which as we recall, are charged neutral]. They reported a peak in the conductance through the nano-wire at zero voltage, which they claimed to be a signature of a spatially separated pair of Majorana fermions forming [6, 7].

The problem with producing quasipartciles which are Majorana fermions is as follows: We don't want electrons to hit the holes, for otherwise they "annihilate". Superconductors paired with topological insulators, or substances that conduct electricity only on their surfaces, can therefore be useful [8]. When the topological insulator meets the superconductor, the electrical field creates a boundary that prevents the electrons from falling into the holes. Furthermore, in superconductors, the distinction between electrons and holes is blurred due to electrons forming *Cooper pairs*, which exhibit boson-like properties. This gives hope to the formation of Majorana fermions. For more detailed discussions on Majorana fermions in solid state physics, as well as how Majorana fermions can contribute to quantum computing, see, e.g., [9]. We nevertheless describe the brief idea: In 2-dimension, there is such thing as *anyon* which can obey statistics ranging continuously between Fermi-Dirac and Bose-Einstein statistics [For the mathematically inclined: this boils down to the fact that for $n \geqslant 3$, we have the fundamental group $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$, but $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$]. Majorana modes, the zero modes [mixtures of particles and holes in equal measure] trapped by Abrikosov vortices [a kind of magnetic flux tubes [11]] in superconductors, have a statistic that is different and even more complex than conventional anyons. The statistics is non-abelian, in the sense that exchanges of particles associated with Majorana modes result not only in a change of the phase of the quantum mechanical wavefunction, but also in the change of the internal states of the modes. As a consequence, while normal fermions have their wavefunction changes from positive to negative with each exchange of position, and hence returning to their original state after two switches, Majorana fermions somehow "remember" their previously taken path [12] [See Fig.1]. This in turn, gives hope that qubits constructed out of Majorana fermion could be more resistant to external influences and thus stable [i.e. they could evade decoherence at the hardware level].

VI. EPILOGUE

What happened to Ettore Majorana is anyone's guess. The hypotheses include for example that, he escaped to a monastery, or that he had been assassinated to prevent his participation in the construction of atomic weapon [13]. On June 7, 2011, Italian media reported that the Carabinieri's RIS had analyzed a photograph of a man

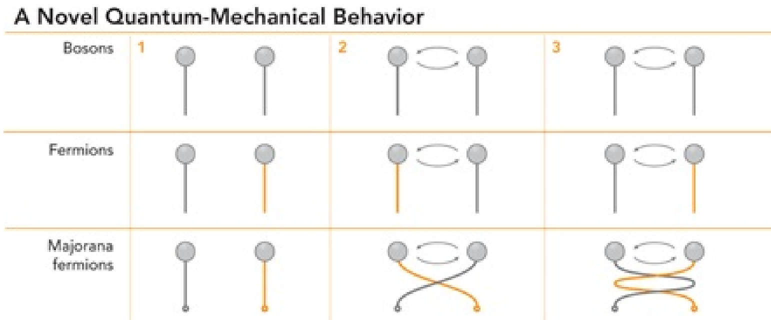

FIG. 1: When two bosons trade places, there is no change in their quantum mechanical state. Normal fermions change the sign of their wavefunction from positive to negative (orange) with each switch, and thus returning to their original state after two switches. Here we imagine the bosons and fermions to be blobs with string dangling below them. Majorana fermion can be thought of as blobs with the strings being taped onto the ground, they "remember" their path [12]. Such property is naturally well investigated by the mathematics of *braid theory* [14].

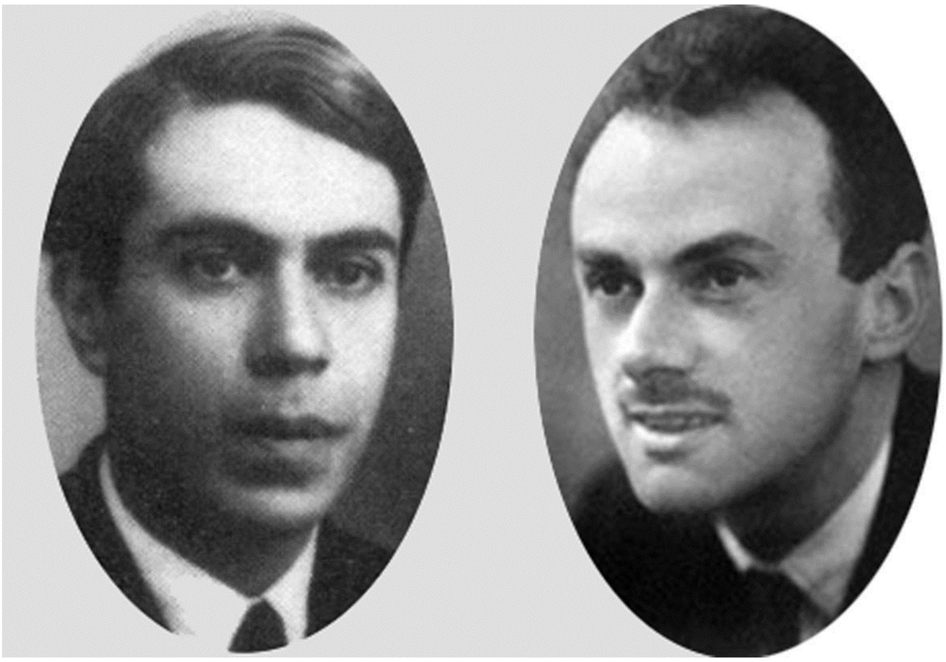

FIG. 2: Ettore Majorana (left) and Paul Dirac (right) proposed two different answers to the question of whether neutrinos and antineutrinos are different particles or a single particle masquerading as two. Nearly a century after neutrinos were proposed, the topic remains an open question. Source: Fermilab [15]

taken in Argentina in 1955, finding ten points of similarity with Majorana's face [10]... The man himself is apparently as elusive, if not more so, than his particle.

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